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Clifford analysis of exterior forms and Fermi–Bose symmetry

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Abstract. Kähler has used the correspondence between the exterior and Clifford algebras to formulate a version of the Dirac equation (the Kähler equation). We exploit the fact that exterior forms may thus be used for the description of half-integer as well as integer spin to present a sypersymmetric model.

The Kähler equation offers an attractive description of half-integer spin in terms of a field which takes values in the exterior algebra of the cotangent space of the space–time manifold (Kähler 1962, Graf 1978, Benn and Tucker 1983a). We call such fields Kähler fields. Since differential forms are natural for the description of integer spin, Kähler's equation offers a certain fusion between fermions and bosons. In a previous letter (Benn and Tucker 1983b) we formulated a supersymmetric model in terms of Kähler fields. We present full details in this paper. We use the conventions of Benn and Tucker (1983a, b, c, 1982).

Kähler's equation requires the use of inhomogeneous differential forms whereas familiar theories of bosons, for example the Maxwell or Klein–Gordon theories, require only homogeneous p-forms. It is therefore instructive to consider a theory of integer spin using inhomogeneous differential forms in a non-trivial way. Lichnerowicz (1964) has shown how the Kemmer–Duffin–Petiau equations can be written using inhomogeneous differential forms. We first compare and contrast this equation with that of Kähler. The massive Kähler equation may be written

$$\mathrm{d}\psi - \delta\psi = m\psi \tag{1}$$

where ψ is a Kähler field and

$$\delta = {*\mathrm{d}*}. \tag{2}$$

We may write the Duffin–Kemmer–Petiau equation in a similar way,

$$\mathrm{d}\phi_+ - \delta\phi_- = m\phi \tag{3}$$

where

$$\phi_\pm = \tfrac{1}{2}(1 \pm \eta)\phi. \tag{4}$$

The automorphism η commutes with S_p, which projects p-form components, and satisfies

$$\eta S_p\phi = (-1)^p S_p\phi. \tag{5}$$

It is readily checked that both (1) and (3) iterate to the Klein–Gordon equation, i.e.

$$\Box\psi = m^2\psi \tag{6}$$

$$\Box\phi = m^2\phi \tag{7}$$

where $\Box$ is the Laplace–Beltrami operator,

$$\Box = -(\mathrm{d}\delta + \delta\mathrm{d}). \tag{8}$$

In this form the Kähler and Duffin–Kemmer–Petiau equations look very similar; to see the crucial difference between them we use the correspondence between the exterior and Clifford algebras (Kähler 1962, Graf 1978, Benn and Tucker 1983a) to rewrite them as

$$e^a{}_\vee\nabla_{X_a}\psi = m\psi \tag{9}$$

$$e^a{}_\vee\nabla_{X_a}\phi + \nabla_{X_a}\phi_\vee e^a = 2m\phi \tag{10}$$

respectively. The $\{e^a\}$ are a basis for the cotangent space over space–time and $\{X_a\}$ are a dual basis. ∇ is the pseudo-Riemannian connection and $_\vee$ denotes the Clifford product. Now in Minkowski space we may find a complete set of covariantly constant minimal rank idempotent projectors $\{P_i\}$, $i = 1, 2, 3, 4$. Right multiplication of (9) by P_i then shows that the Kähler equation decouples into a minimal left ideal on which we may represent Spin (3, 1). The second term in equation (10) prevents the Duffin–Kemmer–Petiau equations from similarly decoupling into ideals. It is convenient to introduce

$$\not{d} = \mathrm{d} - \delta \tag{11}$$

and to define $\not{\not{d}}$ by

$$\overline{\not{d}\phi} = \not{\not{d}}\bar{\phi}.$$

$\bar{\phi} \equiv \xi\eta\phi$ where the anti-automorphism ξ commutes with S_p and satisfies

$$\xi S_p\phi = (-1)^{[p/2]}S_p\phi \tag{12}$$

$[p/2]$ being the integer part of $p/2$. We may then of course write (1) as

$$\not{d}\psi = m\psi \tag{13}$$

and (3) becomes

$$(\not{d} - \not{\not{d}})\phi = 2m\phi. \tag{14}$$

It is of interest to note that for $m \neq 0$ any solution to (14) is also a solution to (13). This is most readily seen by writing (14) as the pair of equations

$$\not{d}\phi = m\phi \qquad (\not{d} + \not{\not{d}})\phi = 0.$$

As a model for supersymmetry we take the action-density four-form

$$\Lambda = S_0(\bar{\psi}_\vee\not{d}\psi_\vee B + \bar{\phi}_\vee\not{d}(\not{d} + \not{\not{d}})\phi)z. \tag{15}$$

ψ and ϕ are real Kähler fields, B is some (as yet unspecified) covariantly constant element of the algebra and z is the volume four-form $*1$.

If, anticipating its semiclassical interpretation, we assign to the real components of ψ elements of a further Grassman algebra, then we set $\varepsilon = +(-1)$ according to whether ψ is taken to have an even (or odd) grading in this algebra. To prevent the action (15) from becoming an exact form we demand $\bar{B} = \varepsilon B$.

The field equations are readily obtained by noting that

$$S_0(\bar{\alpha}_\vee \not{d}\beta)z = \pm S_0(\bar{\beta}_\vee \not{d}\alpha)z \qquad \mathrm{mod}\ \mathrm{d} \tag{16}$$

$$S_0(\bar{\alpha}_\vee \not{\not{d}}\beta)z = \pm S_0(\bar{\beta}_\vee \not{\not{d}}\alpha) \qquad \mathrm{mod}\ \mathrm{d} \tag{17}$$

where + (−) is adopted according to whether the relative grading of α and β in the Grassman algebra is even (odd). With $\bar{B} = \varepsilon B$, the field equations are the massless Kähler equation

$$\not{d}\psi = 0 \tag{18}$$

and

$$\not{d}(\not{d} + \not{\not{d}})\phi = 0. \tag{19}$$

We may recognise ϕ as containing generalised gauge fields by writing (19) as

$$\delta\ \mathrm{d}\phi_- = 0 \tag{20}$$

$$\mathrm{d}\delta\phi_+ = 0. \tag{21}$$

The following are now obviously symmetries:

$$\phi_- \to \phi_- + \mathrm{d}\alpha_+ \tag{22}$$

$$\phi_+ \to \phi_+ + \delta\beta_- \tag{23}$$

where $\alpha_+(\beta_-)$ is any even (odd) element of the algebra. The one-form component of ϕ obviously satisfies the Maxwell equation (for the vector potential). It may be seen that the three-form component is non-propagating by setting $S_3\phi = *C$, for some one-form C, and writing (20) as

$$\mathrm{d}\delta C = 0 \tag{24}$$

or

$$\delta C = \text{constant}. \tag{25}$$

The four-form component of ϕ may be written as the Hodge dual of a 0-form which satisfies the Klein–Gordon equation, whilst the two-form component is an 'anti-symmetric tensor gauge field' (Kemmer 1938, Deser and Witten 1981, Townsend 1981).

It has already been noted that B must satisfy $\bar{B} = \varepsilon B$. If the action is to be diagonal in the minimal left ideals of ψ which are projected with the idempotents $\{P_i\}$ then we further require

$$B\bar{P}_i = P_i B \qquad \forall i = 1, 2, 3, 4. \tag{26}$$

The existence of such a B is guaranteed by the following theorem. (Actually this theorem could have been couched in much more general terms.)

Theorem 1. If $\{P_i\}$ are a complete set of pairwise orthogonal primitive idempotents of the Clifford algebra of an even-dimensional vector space then $\exists B : \bar{P}_i = B^{-1}P_iB\ \forall i$.

Proof. It is trivial that $\bar{P}_i$ has the same rank as P_i. The simplicity of the algebra ensures the existence of a set $\{S_i\}: \bar{P}_i = S_i^{-1}P_iS_i$ (see e.g. Albert 1961). Let $B = \Sigma_i P_iS_i$; then $\Sigma_i P_i = 1 = \bar{1}$ ensures that

$$B^{-1} = \sum_i S_i^{-1}P_i = \sum_i \bar{P}_iS_i^{-1}.$$

Orthogonality of the $\{P_i\}$ gives

$$B^{-1}P_i = S_i^{-1}P_i \qquad \text{and} \qquad P_iB = P_iS_i;$$

thus

$$B^{-1}P_iB = B^{-1}P_iP_iB = S_i^{-1}P_iP_iS_i = S_i^{-1}P_iS_i = \bar{P}_i.$$

Theorem 2. If $\exists S : S\bar{P} = PS$ with $\bar{S} = \pm S$ for some primitive idempotent P, then $\exists\, U : U\bar{Q} = QU$ with $\bar{U} = \pm U$ for any primitive idempotent Q.

Proof. Simplicity of the algebra ensures that $\exists\, T : Q = TPT^{-1}$. So

$$\bar{Q} = \overline{T^{-1}}\,\bar{P}\bar{T} \qquad S\bar{T}\bar{Q} = S\bar{P}\bar{T}$$

$$S\bar{T}\bar{Q} = PS\bar{T} = T^{-1}QTS\bar{T} \qquad TS\bar{T}\bar{Q} = QTS\bar{T}$$

i.e. if $TS\bar{T} \equiv U$

$$U\bar{Q} = QU \qquad \text{and} \qquad \bar{U} = \pm U \text{ if } \bar{S} = \pm S.$$

Corollary. If $\nexists\, B : B\bar{P}_i = P_iB\ \forall i$ with $\bar{B} = B$ $(B \neq 0)$ for some particular set of pairwise orthogonal primitive idempotents $\{P_i\}$, then if $U\bar{Q} = QU$ for any primitive idempotent Q, $\bar{U} = -U$. For suppose we have $U\bar{Q} = QU$; then $Q\bar{U} = \bar{U}\bar{Q}$. If $U + \bar{U} \equiv V$ then

$$V\bar{Q} = QV \qquad \text{and} \qquad \bar{V} = V.$$

By theorem 2 we may find a set of $\{S_i\}$, with $S_i\bar{P}_i = P_iS_i$ and $\bar{S}_i = S_i\ \forall_i$. We now set $B = \Sigma_i P_iS_i$, and have

$$B\bar{P}_i = P_iB\ \forall i \qquad \text{and} \qquad \bar{B} = B.$$

B will only be zero when $V = 0$.

Theorem 1 guarantees the existence of the B in equation (26); to see that such a B must in fact satisfy $\bar{B} = -B$, irrespective of the choice of $\{P_i\}$, it is only necessary to verify explicitly that there does not exist a $B = \bar{B}$ for some arbitrary (conveniently chosen) set of $\{P_i\}$. That is, the action will only be diagonal in the minimal ideals (of the real Clifford algebra) of ψ if it is Grassman odd. This is the case we shall consider.

We define a derivation on the algebra, Q_α, such that

$$Q_\alpha\psi = (\not{d} + \not{\delta})\phi_{\pm\vee}\alpha \tag{27}$$

$$Q_\alpha\phi = (\psi_\vee B_\vee \tilde{\alpha})_\pm. \tag{28}$$

Then $Q_\alpha\Lambda = 0$, mod d, for either choice of signs. α is a covariantly constant, Grassman odd, element of the algebra. We shall further restrict ψ and α by supposing them to lie in a minimal left ideal. The symmetry Q_α then mixes a half-integer spin with an integer spin field. To confirm that this symmetry conforms to what is called supersymmetry, we shall evaluate the commutator of two successive symmetries; this will require the use of the following.

Theorem 3. If $PX = XP = X$, where P is a primitive idempotent of the Clifford algebra of an even-dimensional vector space (over the complex field or the real field with the signature satisfying $\Sigma\,\xi^{aa} = 0, 2 \bmod 8$), then $X = \lambda P$ where λ is a 0-form.

Proof. Suppose we could find a basis for the minimal left ideal projected out by P of the form $\{P, Q_iU_i\}$, $i=1,\ldots,r-1$, $r\equiv 2^{n/2}$, n being the dimension of the underlying vector space, where $U_iP=U_i\ \forall i$ and $\{P, Q_i\}$ are a complete set of pairwise orthogonal primitive idempotents. Then $X=\lambda P+\Sigma_{i=1}^{r-1} C_iQ_iU_i$ for some 0-forms $\{\lambda, C_i\}$. Orthogonality of the idempotents gives $PX=\lambda P$ and so $PX=X\Rightarrow C_i=0\ \forall_i$.

Thus proof of the theorem rests on the existence of a basis of the form used here. We certainly have the set of pairwise orthogonal primitive idempotents $\{P, Q_i\}$, and their orthogonality ensures that the elements in the set $\{P, Q_iU_i\}$ are linearly independent; so it only remains to show that they span the ideal. For a given i we need at least one element $U_i=U_iP$ such that $Q_iU_i\neq 0$. Since the algebra is simple we have an $S_i: Q_i=S_iPS_i^{-1}$. Thus $S_iP\neq 0$ and

$$(S_iP)P=S_iP, \qquad Q_iS_iP=S_iP.$$

Corollary. If $n=4$ we have $S_0P=\frac{1}{4}$ and $\lambda=4S_0X$.

Before we return to the commutator of the symmetries we note how an element of the algebra may be expanded in p-form components. We may choose a homogeneous orthonormal basis for the Clifford algebra, i.e. $\{\Gamma^I\}: g(\Gamma^I, \Gamma^J)=\eta^{IJ}$, where I, J are multi-indices and g is the induced metric on p-forms

$$\begin{aligned} &g(\Gamma^I, \Gamma^J)=S_0(\Gamma^I \vee \xi\Gamma^J) \\ &\eta^{IJ}=0 \qquad\qquad \text{if } I\neq J \\ &\quad\ =-1(+1) \qquad \text{if } I \text{ does (does not) contain a zero.} \end{aligned}$$

(We choose a signature with $g(e^0, e^0)=-1$ and $\Gamma_I=\eta_{IJ}\Gamma^J$, $\eta_{IJ}=\eta^{IJ}$.) Any element of the algebra, ϕ say, may be written $\phi=\Sigma\, a_I\Gamma^I$. Then

$$S_0(\phi_\vee\xi\Gamma_J)=\sum a_IS_0(\Gamma^I{}_\vee\xi\Gamma_J)=\sum a_Ig(\Gamma^I, \Gamma_J)=\sum a_I\delta^I{}_J=a_J.$$

Thus $\phi=\Sigma\, S_0(\phi_\vee\xi\Gamma_I)\Gamma^I$.

To demonstrate how the commutators may be evaluated we choose the second sign in (27) and (28). Then

$$[Q_\alpha, Q_\beta]\psi=(\not{d}+\not{\delta})(\psi B\bar{\alpha})_-\beta-(\alpha\leftrightarrow\beta) \tag{29}$$

where we have omitted the symbols for the Clifford product. In the following we use the normal summation convention. Small Latin indices will be raised and lowered with $\eta^{ab}=\mathrm{diag}(-1,+1,+1,+1)$.

$$\begin{aligned}
[Q_\alpha, Q_\beta]\psi &= e^a\nabla_{X_a}S_1(\psi B\bar{\alpha})\beta+e^a\nabla_{X_a}S_3(\psi B\bar{\alpha})\beta \\
&\qquad -\nabla_{X_a}S_1(\psi B\bar{\alpha})e^a\beta-\nabla_{X_a}S_3(\psi B\bar{\alpha})e^a\beta-(\alpha\leftrightarrow\beta) \\
&= e^aS_1(\nabla_{X_a}\psi B\bar{\alpha})\beta+e^aS_3(\nabla_{X_a}\psi B\bar{\alpha})\beta-S_1(\nabla_{X_a}\psi B\bar{\alpha})e^a\beta \\
&\qquad -S_3(\nabla_{X_a}\psi B\bar{\alpha})e^a\beta-(\alpha\leftrightarrow\beta) \\
&= e^aS_0(\nabla_{X_a}\psi B\bar{\alpha}e_b)e^b\beta-S_0(\nabla_{X_a}\psi B\bar{\alpha}e_b)e^be^a\beta \\
&\qquad +e^aS_0(\nabla_{X_a}\psi B\bar{\alpha}e_bz)e^bz\beta-S_0(\nabla_{X_a}\psi B\bar{\alpha}e_bz)e^bze^a\beta-(\alpha\leftrightarrow\beta) \\
&= 2e^{ab}S_0(\nabla_{X_a}\psi B\bar{\alpha}e_b)\beta+2\eta^{ab}zS_0(\nabla_{X_a}\psi B\bar{\alpha}e_bz)\beta-(\alpha\leftrightarrow\beta).
\end{aligned}$$

Here $e^{ab} \equiv e^a{}_\wedge e^b$. Now if $\alpha P = \alpha$, $\beta P = \beta$

$$[Q_\alpha, Q_\beta]\psi = -2e^{ab}\beta S_0(B\bar{\alpha}e_b\nabla_{X_a}\psi)P - 2\eta^{ab}z\beta S_0(B\bar{\alpha}e_b z\nabla_{X_a}\psi)P - (\alpha \leftrightarrow \beta).$$

Use of theorem 3 now gives

$$\begin{aligned}[Q_\alpha, Q_\beta]\psi &= -\tfrac{1}{2}e^{ab}\beta B\bar{\alpha}e_b\nabla_{X_a}\psi - \tfrac{1}{2}\eta^{ab}z\beta B\bar{\alpha}e_b z\nabla_{X_a}\psi - (\alpha \leftrightarrow \beta)\\
&= -\tfrac{1}{2}e^{ab}S_0(\beta B\bar{\alpha}\xi\Gamma_I)\Gamma^I e_b\nabla_{X_a}\psi - \tfrac{1}{2}zS_0(\beta B\bar{\alpha}\xi\Gamma_I)\Gamma^I e^a z\nabla_{X_a}\psi - (\alpha \leftrightarrow \beta)\\
&= \tfrac{1}{2}e^{ab}S_0(B\bar{\alpha}\xi\Gamma_I\beta - B\bar{\beta}\xi\Gamma_I\alpha)\Gamma^I e_b\nabla_{X_a}\psi + \tfrac{1}{2}zS_0(B\bar{\alpha}\xi\Gamma_I\beta - B\bar{\beta}\xi\Gamma_I\alpha)\Gamma^I e^a z\nabla_{X_a}\psi\\
&= \tfrac{1}{2}e^{ab}S_0[B\bar{\alpha}\xi(\Gamma_I - \bar{\Gamma}_I)\beta]\Gamma^I e_b\nabla_{X_a}\psi + \tfrac{1}{2}zS_0[B\bar{\alpha}\xi(\Gamma_I - \bar{\Gamma}_I)\beta]\Gamma^I e^a z\nabla_{X_a}\psi\end{aligned}$$

where we have used $S_0X = S_0\bar{X}$, $\bar{B} = -B$ and the fact that α and β are Grassman odd. So only the anti-conjugate Γ_I enter into the expansion; namely the one-forms and two-forms. Thus

$$\begin{aligned}[Q_\alpha, Q_\beta]\psi &= e^{ab}S_0(B\bar{\alpha}e_c\beta)e^c e_b\nabla_{X_a}\psi + zS_0(B\bar{\alpha}e_c\beta)e^c e^a z\nabla_{X_a}\psi\\
&\quad - \tfrac{1}{2}e^{ab}S_0(B\bar{\alpha}e_{cd}\beta)e^{cd}e_b\nabla_{X_a}\psi - \tfrac{1}{2}zS_0(B\bar{\alpha}e_{cd}\beta)e^{cd}e^a z\nabla_{X_a}\psi\\
&= S_0(B\bar{\alpha}e_c\beta)(e^{ab}e^c e_b + ze^c e^a z)\nabla_{X_a}\psi - \tfrac{1}{2}S_0(B\bar{\alpha}e_{cd}\beta)\\
&\quad \times (e^{ab}e^{cd}e_b + ze^{cd}e^a z)\nabla_{X_a}\psi\\
&= S_0(B\bar{\alpha}e_c\beta)(e^a e^b e^c e_b - 2e^c e^a)\nabla_{X_a}\psi - \tfrac{1}{2}S_0(B\bar{\alpha}e_{cd}\beta)(e^a e^b e^{cd}e_b)\nabla_{X_a}\psi.\end{aligned}$$

To complete the calculation we observe

$$e^b S_p X e_b = (4-2p)\eta S_p X \tag{30}$$

and obtain

$$[Q_\alpha, Q_\beta]\psi = -4S_0(B\bar{\alpha}e^a\beta)\nabla_{X_a}\psi. \tag{31}$$

A similar calculation shows that we obtain the same result for a different choice of signs in (27) and (28). If

$$K = -4S_0(B\bar{\alpha}e^a\beta)X_a \tag{32}$$

then K is covariantly constant. This ensures that K is a translational Killing vector. For any Killing vector we have

$$\mathcal{L}_K\phi = \nabla_K\phi + \tfrac{1}{4}(\mathrm{d}\tilde{K}_\vee\phi - \phi_\vee\,\mathrm{d}\tilde{K}) \tag{33}$$

where $\mathcal{L}_K$ denotes the Lie derivative. The metric dual, $\tilde{K}$, is defined by

$$\tilde{K}(X) = g(K, X) \qquad \forall X \in TM. \tag{34}$$

Any covariantly constant vector has a closed metric dual ($\mathrm{d}\tilde{K} = 0$) and thus (33) enables (31) to be written

$$[Q_\alpha, Q_\beta]\psi = \mathcal{L}_K\psi. \tag{35}$$

We may evaluate the commutator of the symmetry on ϕ as

$$[Q_\alpha, Q_\beta]\phi_\pm = \{(\not{d} + \not{\delta})\phi_\pm(\alpha B\bar{\beta} - \beta B\bar{\alpha})\}_\pm. \tag{36}$$

Now $\overline{\alpha B\bar{\beta}} = -\beta\bar{B}\bar{\alpha} = \beta B\bar{\alpha}$ so only the anti-conjugate part of $\alpha B\bar{\beta}$ enters into the commutator; that is, the one-form and two-form components. Thus

$$[Q_\alpha, Q_\beta]\phi_\pm = \{(\not{d} + \not{\delta})\phi_\pm 2S_1(\alpha B\bar{\beta})\}_\pm + \{(\not{d} + \not{\delta})\phi_\pm 2S_2(\alpha B\bar{\beta})\}_\pm. \tag{37}$$

Since $\eta(\not{d}+\not{\delta}) = -(\not{d}+\not{\delta})\eta$ and $\eta S_2 = S_2$ the second term vanishes. Using the covariant constancy of α, B, β we write (37) as

$$[Q_\alpha, Q_\beta]\phi_\pm = (\not{d}-\not{\delta})[2\phi_\pm S_1(\alpha B\bar{\beta})] + \not{\delta}[2\phi_\pm S_1(\alpha B\bar{\beta})] + \not{\delta}\phi_\pm 2S_1(\alpha B\bar{\beta}).$$

Recalling the relations used to write (3) as (14), and (9) as (13)

$$\begin{aligned}[Q_\alpha, Q_\beta]\phi_\pm &= \mathrm{d}[4\phi_\pm S_1(\alpha B\bar{\beta})]_+ - \delta[4\phi_\pm S_1(\alpha B\bar{\beta})]_- - 2\nabla_{X_a}\phi[S_1(\alpha B\bar{\beta})e^a + e^a S_1(\alpha B\bar{\beta})]\\ &= \mathrm{d}[4\phi_\pm S_1(\alpha B\bar{\beta})]_+ - \delta[4\phi_\pm S_1(\alpha B\bar{\beta})]_- - 4S_0(\alpha B\bar{\beta}e^a)\nabla_{X_a}\phi\end{aligned}$$

$$[Q_\alpha, Q_\beta]\phi_\pm = \mathscr{L}_K\phi + \mathrm{d}[4\phi_\pm S_1(\alpha B\bar{\beta})]_+ - \delta[4\phi_\pm S_1(\alpha B\bar{\beta})]_-. \tag{38}$$

Reference to equations (22) and (23) shows that, for either choice of signs, the second (non-vanishing) term is a gauge transformation.

We have demonstrated how, with the appropriate restriction on the parameters, the Q_α generate a supersymmetry, where the commutator closes on a translation off-shell. For the first choice of signs in our symmetry we mix a spinor with the four-form and two-form components of ϕ, both of which are propagating. We thus recover a model presented by Siegel (1979) with the scalar replaced by a pseudo-scalar. (Had we chosen the operator $\not{d}-\not{\delta}$ we would have obtained a scalar.) The second choice of sign mixes a spinor with the one-form and three-form components of ϕ. The three-form component of ϕ is non-propagating and we observe a version of a supersymmetric Maxwell–Majorana neutrino model where the auxilary field is encoded into an 'anti-symmetric tensor gauge field'.

It is satisfying to see how the 'bosonic' Kähler field contains the necessary degrees of freedom for off-shell closure of supersymmetry in this model; and instructive to be able to calculate free from a reliance on matrix representations of the algebra. However, in this formalism the supersymmetry appears rather superficial. We are able to give the Kähler equation a spinorial interpretation by relying on the covariantly constant minimal rank projectors which exist in Minkowski space. However, it would appear that in an arbitrary space we do not have such projectors (although in our minds the important question of the existence of covariantly constant projectors in arbitrary space–times requires a definite answer). If this is the case, and we wish to take Kähler's equation seriously, then rather than a symmetry which mixes half-integer and integer spin fields in an arbitrary space we are led to a more profound fusion between such fields.

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